

# UNITARITY CONSTRAINTS ON NEUTRINO MASS AND MIXINGS

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We explore the implications of imposing the constraint that two neutrino flavors (which for definiteness we take to be  $\nu_\mu$  and  $\nu_\tau$ ) are similarly coupled to the mass basis in addition to the unitarity constraints. Implications of this scheme for specific experimental situations are discussed.

## 1 Introduction

Recent observations of atmospheric neutrinos and especially their zenith-angle dependence<sup>1</sup>, strongly suggest that muon neutrinos maximally mix with the tau neutrinos. Motivated by this observation and recent theoretical work on neutrino mass models<sup>2</sup>, we explored<sup>3</sup> the implications of imposing the constraint that two neutrino flavors (which for definiteness we take to be  $\nu_\mu$  and  $\nu_\tau$ ) are similarly coupled to the mass basis in addition to the unitarity constraints.

Although the invisible width of the Z particle constraints the number of active neutrino flavors to be three, it is nevertheless worthwhile to consider the possibility of the existence of sterile neutrino states for a number of reasons: i) The possibility of oscillation of atmospheric muon neutrinos into sterile states is not completely ruled out. ii) If the LSND results<sup>4</sup> are confirmed, since the analysis of LSND, atmospheric<sup>1</sup> and solar<sup>5</sup> neutrinos point out to different mass scales, one needs to introduce sterile neutrinos. iii) Serious problems such as the abundance of alpha particles that arise when core-collapse supernovae with neutrino-driven wind are considered as sites of r-process nucleosynthesis can be avoided by the oscillations of active neutrinos into sterile ones<sup>6,7</sup>. Even though cosmological and astrophysical bounds rule out heavier sterile states<sup>8</sup>, the effect of the lighter sterile neutrinos on big-bang nucleosynthesis is controversial<sup>9</sup>.

Hence we consider three active flavors and an arbitrary number (which could be taken to be zero) of sterile neutrinos. The  $N \times N$  neutrino mixing

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matrix will be denoted by  $U_{\alpha i}$  where  $\alpha$  denotes the flavor index and  $i$  denotes the mass index:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle. \quad (1)$$

We impose the constraint that  $U_{\mu i}$  and  $U_{\tau i}$  are proportional for all but one mass eigenstate, which we choose for definiteness to be the third mass eigenstate:

$$U_{\mu i} \sim U_{\tau i} \neq 0, \forall i \neq 3. \quad (2)$$

We write this condition in terms of an arbitrary angle  $\phi$  and an arbitrary phase  $\eta$ :

$$\sin \phi U_{\mu i} = e^{i\eta} \cos \phi U_{\tau i} \neq 0, \forall i \neq 3. \quad (3)$$

Note that, in our formalism, we permit CP-violating phases. Introducing the quantity

$$A = \sum_{i \neq 3} [|U_{\mu i}|^2 + |U_{\tau i}|^2] \quad (4)$$

and using Eq. (3) along with the unitarity of the mixing matrix one can easily show that<sup>3</sup>

$$A = 1, \quad (5)$$

$$U_{\mu 3} = -\sin \phi e^{i\delta} e^{i\eta}, \quad (6)$$

$$U_{\tau 3} = \cos \phi e^{i\delta}, \quad (7)$$

where  $\delta$  is a phase to be determined, and

$$U_{\alpha 3} = 0, \quad \alpha \neq \mu, \tau, \quad (8)$$

Introducing the states

$$|\tilde{\nu}_\mu\rangle = \cos \phi |\nu_\mu\rangle + \sin \phi e^{i\eta} |\nu_\tau\rangle, \quad (9)$$

and

$$|\tilde{\nu}_\tau\rangle = -\sin \phi e^{-i\eta} |\nu_\mu\rangle + \cos \phi |\nu_\tau\rangle, \quad (10)$$

It follows that

$$|\tilde{\nu}_\mu\rangle = \frac{1}{\cos \phi} \sum_{i \neq 3} U_{\mu i} |\nu_i\rangle, \quad (11)$$

$$|\tilde{\nu}_\tau\rangle = e^{i\delta} |\nu_3\rangle. \quad (12)$$

and

$$|\nu_\alpha\rangle = \sum_{i \neq 3} U_{\alpha i} |\nu_i\rangle, \quad \alpha \neq \mu, \tau. \quad (13)$$

This is a remarkable result which simply follows from the assumption of Eq. (3). This assumption leads to a decoupling of all the other flavors from the chosen (the third in our choice) mass eigenstate in the neutrino mixing matrix.

### *Three Active Flavors*

For three active flavors we get

$$\begin{pmatrix} |\nu_e\rangle \\ |\tilde{\nu}_\mu\rangle \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu1}/\cos\phi & U_{\mu2}/\cos\phi \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}. \quad (14)$$

The solar neutrino data in this case could be explained by either the matter-enhanced or vacuum  $\nu_e \rightarrow \tilde{\nu}_\mu$  oscillations.

In the special case of  $\phi = \pi/4$ , the full mixing matrix is given by<sup>10</sup>

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sqrt{2}\sin\theta & \sqrt{2}\cos\theta & \frac{1}{\sqrt{2}}e^{i\delta} \\ \sqrt{2}\sin\theta & \sqrt{2}\cos\theta & -\frac{1}{\sqrt{2}}e^{i\delta} \end{pmatrix}. \quad (15)$$

The limiting case of  $\theta = \pi/4$  and  $\delta = 0$  yields bi-maximal mixing of three active neutrinos<sup>11,12,13</sup>.

### *An Arbitrary Number of Flavors*

In general  $N$  flavors mix with the fundamental representation of  $U(N)$ . An arbitrary  $U(N)$  element can be written as a product of  $N(N-1)/2$  different non-commuting  $SU(2)$  rotations and a diagonal matrix:

$$U_{i\alpha}^\dagger = R_{12}R_{13}R_{14}\cdots R_{23}R_{24}\cdots \begin{pmatrix} e^{i\delta_1} & 0 & 0 & \cdot \\ 0 & e^{i\delta_2} & 0 & \cdot \\ 0 & 0 & e^{i\delta_3} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad (16)$$

where e.g.

$$R_{14} = \begin{pmatrix} C_{14} & 0 & 0 & S_{14}^* & \cdot \\ 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ -S_{14} & 0 & 0 & C_{14}^* & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \quad (17)$$

Our choice of parameters leads to

$$\begin{aligned} C_{\alpha 3} &= 1, \forall \alpha \neq 2 \\ C_{23} &= \cos \phi \\ S_{23} &= e^{i\eta} \sin \phi, \end{aligned} \quad (18)$$

hence our choice reduces the number of parameters from  $N(N-1)/2$  to  $(N^2 - 3N + 4)/2$ .

## 2 Specific Cases

Here we summarize implications of our scheme for three different experimental situations.

### 2.1 Atmospheric Neutrinos

If we have only active neutrinos with  $m_1 \sim m_2$  we have the standard result:

$$P(\nu_\mu \rightarrow \nu_\tau) = \sin^2 2\phi \sin^2 \left[ \frac{(m_3^2 - m_2^2)L}{4E} \right] \quad (19)$$

If we have only one sterile state in addition to the active neutrinos and there is the mass hierarchy  $m_4 > m_3 \gg m_2 > m_1$  (where  $m_2^2 - m_1^2$  is of order of the solar neutrino solution) we get the following result for the  $\nu_\mu \rightarrow \nu_\tau$  conversion probability:

$$\begin{aligned} P(\nu_\mu \rightarrow \nu_\tau) &= \sin^2 2\phi \sin^2 \left[ \frac{(m_3^2 - m_2^2)L}{4E} \right] + 4|U_{\mu 4}|^4 \tan^2 \phi \sin^2 \left[ \frac{(m_4^2 - m_2^2)L}{4E} \right] \\ &- 8 \sin^2 \phi |U_{\mu 4}|^2 \sin \left[ \frac{(m_4^2 - m_2^2)L}{4E} \right] \sin \left[ \frac{(m_3^2 - m_2^2)L}{4E} \right] \cos \left[ \frac{(m_4^2 - m_3^2)L}{2E} \right]. \end{aligned} \quad (20)$$

It will be instructive to do a fit to the SuperKamiokande atmospheric neutrino data with Eq. (20).

### 2.2 Reactor Neutrinos

In our scheme, if the value of  $(m_2^2 - m_1^2)$  is determined from the solar neutrino data, for reactor neutrino experiments we can assume  $(m_2^2 - m_1^2) \ll E/L$ . We then have

$$P(\nu_e \rightarrow \nu_e) = 1 - \sin^2 2\theta_{\text{eff}} \sin^2 \left[ \frac{(m_4^2 - m_1^2)L}{4E} \right], \quad (21)$$

where

$$\sin^2 2\theta_{\text{eff}} = 4|U_{e4}|^2(1 - |U_{e4}|^2). \quad (22)$$

For the large values of  $(m_4^2 - m_1^2)$  that would help the r-process nucleosynthesis in the neutrino-driven wind models of supernova CHOOZ experiment gives a bound of  $|U_{e4}|^2 < 0.047$ <sup>14</sup>. The best limit,  $|U_{e4}|^2 < 0.005$  comes from the BUGEY experiment<sup>15</sup> and is still consistent with the conversion into sterile neutrinos in supernovae<sup>6</sup>.

### 2.3 Neutrinoless Double Beta Decay

The current data indicates

$$\mathcal{M}_{ee} = \sum_i m_i |U_{ei}|^2 < \sim 0.5 \text{eV}. \quad (23)$$

In our scheme  $\mathcal{M}_{ee} = \sum_{i \neq 3} m_i |U_{ei}|^2$ . Thus for three flavors  $\mathcal{M}_{ee}$  depends only on  $m_1$  and  $m_2$ , not on  $m_3$ . It is possible to enforce  $\mathcal{M}_{ee} \equiv 0$  for bi-maximal mixing<sup>11,13</sup>. When sterile neutrinos are included this puts a limit on  $m_4$ . One should emphasize that the uncertainties of the nuclear matrix elements could be rather large so it may not be necessary to impose  $\mathcal{M}_{ee} \equiv 0$ .

## 3 Conclusions

We explored the implications of imposing the constraint that two neutrino flavors (which for definiteness we take to be  $\nu_\mu$  and  $\nu_\tau$ ) are similarly coupled to the mass basis in addition to the unitarity constraints. We allow three active and an arbitrary number of sterile neutrinos. We show that in this scheme one of the mass eigenstates decouples from the problem, reducing the dimension of the flavor space by one. This result allows significant simplification in the treatment of matter-enhanced neutrino transformation where multiple flavors and level crossings are involved.

When the constraint of Eq. (18) is imposed, which was motivated by the recent experimental results at Superkamiokande, the form of Eq. (16) indicates the existence of a coset structure of the neutrino mixing matrix. Recent related work discussed the existence of a an  $Sp(4)$  symmetry in the neutrino mass sector<sup>16</sup>. It was shown that the most general neutrino mass Hamiltonian sits in the  $Sp(4)/SU(2) \times U(1)$  coset space where  $U(1)$  is the chirality transformation and the  $SU(2)$  generates the see-saw transformation. At the moment it is not clear what the relation, if any, between these two coset structures is.

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